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WIGNER'S THEOREM REVISITED

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Abstract. In this paper, we give the general solution of the functional equation

$$\big\{\|f(x)+f(y)\|, \|f(x)-f(y)\|\big\} = \big\{\|x+y\|, \|x-y\|\big\} \qquad (x, y \in X)$$

where $f : X \to Y$ and X, Y are inner product spaces. Related equations are also considered. Our main tool is a real version of Wigner's unitary–antiunitary theorem.

1. Introduction

An isometry from a normed space X into another normed space Y is a function $f : X \to Y$ which satisfies the equality

$$\|f(x) - f(y)\| = \|x - y\| \qquad (x, y \in X). \tag{1}$$

This equation implies strong structural properties for the function f. A classical result in this direction is a celebrated theorem of Mazur and Ulam [6] which states that an isometry f of a real normed space *onto* another normed space is necessarily affine. In other words, for the surjective solutions $f : X \to Y$ of (1), $x \mapsto f(x) - f(0)$, is a norm preserving linear map. Baker [2] showed that the same conclusion remains valid if the surjectivity assumption is replaced by the strict convexity of the target space Y. Another important result which is related to linear isometries is Wigner's theorem [16] and its generalization obtained by Rätz [14, Corollary 8(a)]. For further generalizations of this fundamental result, we mention the papers [1], [3], [5], [7], [8], [9], [10], [11], [12], [13], and [15].

Assuming that X and Y are *real* inner product spaces, Rätz's result characterizes functions $f : X \to Y$ that are phase equivalent to a linear isometry (i.e., there exists a function $\varepsilon : X \to \{-1, 1\}$ such that εf is a norm preserving real linear map) by the property

$$|\langle f(x), f(y)\rangle| = |\langle x, y\rangle| \qquad (x, y \in X). \tag{2}$$

In the complex setting, Wigner's theorem [16] (cf. also [5]) says that the solutions of (2) are phase equivalent to a linear or conjugate linear isometry. Without assuming that X and Y are real inner product spaces, we can easily see that all functions $f : X \to Y$ that

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are phase equivalent to a real linear isometry are also solutions of the functional equation

$$\big\{\|f(x)+f(y)\|, \|f(x)-f(y)\|\big\} = \big\{\|x+y\|, \|x-y\|\big\} \qquad (x,y \in X). \tag{3}$$

Indeed, if $\varepsilon : X \to \{-1,1\}$ and $g := \varepsilon f$ is a norm preserving real linear map, then, for all $x, y \in X$,

$$\begin{aligned}\|f(x) \pm f(y)\| &= \|\varepsilon(x)g(x) \pm \varepsilon(y)g(y)\| = \|g(x) \pm \varepsilon(x)\varepsilon(y)g(y)\| \\ &= \|g(x \pm \varepsilon(x)\varepsilon(y)y)\| = \|x \pm \varepsilon(x)\varepsilon(y)y\|,\end{aligned}$$

which implies (3) because $\varepsilon(x)\varepsilon(y)$ is either equal to 1 or to -1.

The aim of this short note is to show that the converse also holds provided that X, Y are inner product spaces. That is, in that case, all solutions $f : X \to Y$ of (3) are phase equivalent to a real linear isometry. The main tool in the proof is Rätz's characterization theorem described above.

2. The equivalence of some functional equations related to (3) and our main results

Throughout the remaining part of this paper, X and Y denote real or complex inner product spaces. We note that every complex linear space is trivially a real linear space and if $\langle\cdot,\cdot\rangle$ is a complex inner product on X (or on Y) then $\langle\langle\cdot,\cdot\rangle\rangle$ defined as $\langle\langle x, y\rangle\rangle = \Re\langle x, y\rangle$ is real inner product on X which induces the same norm. (Here $\Re z$ stands for the real part of the complex number z.) Therefore, we may assume that $\langle\cdot,\cdot\rangle$ always denotes the real inner product on X and Y. A function $f : X \to Y$ is called real linear if f is additive and homogeneous with respect to real numbers. Real linearity does not imply complex linearity in general as it is shown by the following example constructed by Rätz [14]: Let $X = Y = \mathbb{C}^2$ equipped with the usual inner product

$$\langle(x_1,x_2),(y_1,y_2)\rangle = x_1\overline{y}_1 + x_2\overline{y}_2 \qquad ((x_1,x_2),(y_1,y_2) \in \mathbb{C}^2),$$

and $f(x_1,x_2) = (x_1,\overline{x}_2)$ for $(x_1,x_2) \in \mathbb{C}^2$. An easy calculation shows that f is norm preserving real linear, but it is not complex-homogeneous, and hence it is not linear.

We begin with a characterization of norm-preserving real linear maps between inner product spaces.

Theorem 1. *For any $f : X \to Y$, the following three statements are equivalent:*
- (i) $\|f(x)+f(y)\| = \|x+y\| \quad (x,y \in X)$*;*
- (ii) $\langle f(x), f(y)\rangle = \langle x, y\rangle \quad (x,y \in X)$*;*
- (iii) *f is a norm-preserving real linear map.*

Proof. Suppose first that (i) holds. Putting $x = y$, it follows that f is norm-preserving. Now using (i) and the norm preserving property, we get

$$2\langle f(x), f(y)\rangle = \|f(x)+f(y)\|^2 - \|f(x)\|^2 - \|f(y)\|^2 = \|x+y\|^2 - \|x\|^2 - \|y\|^2 = 2\langle x, y\rangle,$$

which proves (ii).

Now suppose (ii). Putting $x = y$, the norm preserving property of f follows. Using (ii) three times, for all $x, y, z \in X$, we obtain

$$\langle f(x+y) - f(x) - f(y), f(z)\rangle = \langle x+y, z\rangle - \langle x, z\rangle - \langle y, z\rangle = 0.$$

Applying this identity for $z \in \{x+y, x, y\}$, we get

$$\langle f(x+y) - f(x) - f(y), f(x+y) - f(x) - f(y) \rangle = 0,$$

which yields that f is additive.

Finally, assume that f is a norm-preserving real linear map. Then, by the additivity and the norm-preserving property, we get that $\|f(x) + f(y)\| = \|f(x+y)\| = \|x+y\|$ which implies (i). □

Remark. The equivalence of (i) and (iii) can easily be proved by supposing only that X and Y are normed spaces and Y is strictly convex. Indeed, the substitution $y = x$ in (i) implies that f is norm-preserving. Therefore $\|f(x) + f(y)\| = \|f(x+y)\|$ holds for all $x, y \in X$. Applying a result of Ger [4], we obtain that f is additive which implies (iii). On the other hand, (i) follows from (iii) immediately.

In the following theorem, we list four equivalent conditions that are equivalent to (3).

Theorem 2. *For any $f : X \to Y$, the following five statements are equivalent:*

(i) (3) *holds;*
(ii) $\|f(x) + f(y)\| + \|f(x) - f(y)\| = \|x+y\| + \|x-y\| \quad (x, y \in X)$*;*
(iii) $f(0) = 0$ *and* $\|f(x) + f(y)\|\|f(x) - f(y)\| = \|x+y\|\|x-y\| \quad (x, y \in X)$*;*
(iv) $|\langle f(x), f(y) \rangle| = |\langle x, y \rangle| \quad (x, y \in X)$*;*
(v) *There exists a function $\varepsilon : X \to \{-1, 1\}$ such that εf is a norm-preserving real linear map.*

Proof. The statement (i) implies (ii) obviously. With the substitution $y = x$, it follows from (ii) that f is norm preserving, i.e.,

$$\|f(x)\| = \|x\| \qquad (x \in X). \tag{4}$$

With $x = 0$, this yields $f(0) = 0$. Now we square the equation in (ii) to obtain

$$\begin{aligned} \|f(x)\|^2 &+ 2\langle f(x), f(y) \rangle + \|f(y)\|^2 + 2\|f(x) + f(y)\|\|f(x) - f(y)\| \\ &+ \|f(x)\|^2 - 2\langle f(x), f(y) \rangle + \|f(y)\|^2 \\ &= \|x\|^2 + 2\langle x, y \rangle + \|y\|^2 + 2\|x+y\|\|x-y\| + \|x\|^2 - 2\langle x, y \rangle + \|y\|^2. \end{aligned}$$

Using (4), the above equality simplifies to the second equality in (iii). Thus (ii) implies (iii).

Substituting $y = 0$ into the second equation in (iii), we get (4). Squaring the second equation in (iii) and using (4) again, we obtain that

$$\begin{aligned} &\big(\|x\|^2 + 2\langle f(x), f(y) \rangle + \|y\|^2\big)\big(\|x\|^2 - 2\langle f(x), f(y) \rangle + \|y\|^2\big) \\ &\qquad = \big(\|x\|^2 + 2\langle x, y \rangle + \|y\|^2\big)\big(\|x\|^2 - 2\langle x, y \rangle + \|y\|^2\big). \end{aligned}$$

This simplifies to

$$\big(\langle f(x), f(y) \rangle\big)^2 = \big(\langle x, y \rangle\big)^2 \qquad (x, y \in X),$$

which is equivalent to the equation in (iv) proving that (iii) implies (iv).

If (iv) holds, then, by the result of Rätz [14, Corollary 8(a)] described in the introduction, (v) follows.

Finally, (v) implies (i) as we have seen it in the introduction. □

The following corollary describes the continuous solutions of (3).

Corollary 3. *Let X be at least two dimensional. For a continuous function $f : X \to Y$, the four equivalent statements (i)–(iv) of Theorem 2 hold if and only if f is a norm-preserving real linear map.*

Proof. Assume that f is a continuous function satisfying any of the conditions (i)–(iv) of Theorem 2. Then there exists a function $\varepsilon : X \to \{-1, 1\}$ such that εf is norm-preserving and real linear. Thus, by Theorem 1,

$$\varepsilon(x)\varepsilon(y)\langle f(x), f(y)\rangle = \langle x, y\rangle \qquad (x, y \in X).$$

If $y \neq 0$, then there exists an open ball U around y such that

$$\varepsilon(x) = \varepsilon(y)\frac{\langle x, y\rangle}{\langle f(x), f(y)\rangle} \qquad (x \in U).$$

This, by the continuity of f, shows that ε is continuous on U and hence it is constant on U. The set $X \setminus \{0\}$ is connected (because X is at least two dimensional), therefore ε is constant on $X \setminus \{0\}$. Thus f must be a norm-preserving real linear map. □

Remark. In the exceptional nontrivial case when X is one dimensional and real, say $X = \{\lambda a : \lambda \in \mathbb{R}\}$ with some $a \in X$, $\|a\| = 1$, and Y is at least one dimensional, the above argument shows that ε is constant on the set of positive reals and constant also on the set of negative reals. Therefore f is either a norm-preserving real linear map or $f(\lambda a) = |\lambda| b$ for all $\lambda \in \mathbb{R}$ and for some $b \in Y$ with $\|b\| = 1$.

Finally, we formulate two open problems.

Problem 1. What are the solutions $f : X \to Y$ of (3) when X and Y are normed but not necessarily inner product spaces? Under what conditions does it remain valid that, for the solutions of (3), εf is real linear for some function $\varepsilon : X \to \{-1, 1\}$?

Problem 2. Let X and Y be complex normed spaces. Let n be a fixed positive integer and denote $\beta_1, \ldots, \beta_n$ the nth roots of unity. These elements form a multiplicative subgroup of the unit circle in $\mathbb{C}$. Find the solutions $f : X \to Y$ of the following generalization of (3):

$$\big\{\|f(x) - \beta_k f(y)\| : k \in \{1, \ldots, n\}\big\} = \big\{\|x - \beta_k y\| : k \in \{1, \ldots, n\}\big\} \qquad (x, y \in X). \quad (5)$$

Obviously, this is the isometry equation in case $n = 1$, and the case $n = 2$ was just discussed in this paper. One can also see that if there exists a function $\varepsilon : X \to \{\beta_1, \ldots, \beta_n\}$ such that εf is complex linear and norm-preserving, then f satisfies (5). Under what conditions does it remain valid that, for the solutions of (5), εf is complex linear and norm-preserving for some function $\varepsilon : X \to \{\beta_1, \ldots, \beta_n\}$?

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